HANS DUISTERMAAT'S CONTRIBUTIONS TO POISSON GEOMETRY

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Abstract. Hans Duistermaat was scheduled to lecture in the 2010 School on Poisson Geometry at IMPA, but passed away suddenly. This is a record of a talk I gave at the 2010 Conference on Poisson Geometry (the week after the School) to share some of my memories of him and to give a brief assessment of his impact on the subject.

Johannes Jisse (Hans) Duistermaat (1942–2010) earned his doctorate in 1968 at the University of Utrecht under the direction of Hans Freudenthal. After holding a postdoctoral position at the University of Lund, a professorship at the University of Nijmegen, and a visiting position at the Courant Institute, he returned to Utrecht in 1975 to take over Freudenthal's chair after the latter's retirement. He held this chair until his own retirement in 2007. He continued to play a role in the life of the Utrecht Mathematical Institute and kept up his mathematical activities until he was struck down in March 2010 by a case of pneumonia contracted while on chemotherapy for cancer.

I got to know Hans Duistermaat as an undergraduate when I took his freshman analysis course at Utrecht. At that time he taught the course from lecture notes that were in style and content close to Dieudonné's book [23]. Although this was a smashing success with some students, I think Hans realized he had to reduce the potency so as not to leave behind quite so many of us, and over the years the lecture notes grew into his and Joop Kolk's still very substantial undergraduate textbook [31]. Anyway, I quickly became hooked and realized that I wanted one day to become his graduate student. This came to pass and I finished in 1990 my thesis work on a combination of two of Hans' favourite topics, Lie groups and symplectic geometry.

The online Mathematics Genealogy Project (accessed 19 December 2010) lists twenty-four PhD students under Duistermaat's name. What the website does not tell you is that his true adviser was not Freudenthal, but the applied mathematician G. Braun, who died one year before Hans' thesis work was finished and about whom I have been able to find little information.

The subject of our conference, Poisson geometry, is now firmly ensconced on both sides of the Atlantic as well as on both sides of the equator, as Alan Weinstein observed this week, and Henrique Bursztyn has kindly asked me to speak about Hans Duistermaat's impact on the field. In a narrow sense Duistermaat contributed very little to Poisson geometry. The subject dearest to his heart was differential equations, although he had an unusual geometric intuition. As far as I know (thanks to Rui Loja Fernandes), the notion of a *Poisson manifold* appears just once in his written work, namely in a book [26] on discrete dynamical systems on elliptic surfaces, which he finished not long before his death and which

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has just been published. But Poisson *brackets* can be found in most of his papers, and the fact is that he has contributed many original ideas to the area.

Bispectral problem for Schrödinger equations. For instance, his paper [27] with Alberto Grünbaum continues to be influential in the literature on integrable systems and noncommutative algebraic geometry. It contains the solution of the at first rather strange-sounding bispectral problem, for what potentials $V(x)$ do the solutions $f(x,\lambda)$ of the equation $-f'' + Vf = \lambda f$ satisfy a differential equation in the spectral parameter λ? A comment on the bibliography of this paper: most mathematicians disregard the classics, but Hans was never afraid to go back to the sources. He was widely read in the older literature on analysis and differential geometry, and used it to great effect in his own writings.

Resonances. To mention a lesser-cited paper [25], let me remind you of Duistermaat's insight how resonances in a Hamiltonian system may preclude complete integrability, as explained to us earlier this week by Nguyen Tien Zung.

Lie III. Sometimes the flow of ideas was remarkably indirect. His and Joop Kolk's book on Lie groups [30] (which, despite having appeared in Springer's Universitext series, is not exactly an elementary graduate text) is much closer to Lie's original point of view pertaining to differential equations than modern treatments such as Bourbaki [9], which are more algebraic in spirit. Nevertheless the book is notable for several innovations, particularly its proof of Lie's third fundamental theorem in global form, which I think deserves to become the standard argument and which runs in outline as follows.

The thing to be proved is that for every finite-dimensional real Lie algebra $\mathfrak{g}$ there exists a simply connected finite-dimensional real Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$. Choose a norm on $\mathfrak{g}$. Then the space $P(\mathfrak{g})$ of continuous paths $[0,1] \to \mathfrak{g}$ equipped with the supremum norm is a Banach space. For each γ in $P(\mathfrak{g})$ let A_γ be the continuously differentiable path of linear endomorphisms of $\mathfrak{g}$ determined by the linear initial-value problem

$$A'_\gamma(t) = \mathrm{ad}(\gamma(t)) \circ A_\gamma(t), \qquad A_\gamma(0) = \mathrm{id}_{\mathfrak{g}}\,.$$

Duistermaat and Kolk prove that the multiplication law

$$(\gamma \cdot \delta)(t) = \gamma(t) + A_\gamma(t)\delta(t)$$

turns $P(\mathfrak{g})$ into a Banach Lie group. Next they introduce a subset $P(\mathfrak{g})_0$, which consists of all paths γ that can be connected to the constant path 0 by a family of paths γ_s which is continuously differentiable with respect to s and has the property that

$$\int_0^1 A_{\gamma_s}(t)^{-1}\frac{\partial \gamma_s}{\partial s}(t)\,dt = 0$$

for $0 \le s \le 1$. The subset $P(\mathfrak{g})_0$ is a closed connected normal Banach Lie subgroup of $P(\mathfrak{g})$ of finite codimension, and the quotient $P(\mathfrak{g})/P(\mathfrak{g})_0$ is a simply connected Lie group with Lie algebra $\mathfrak{g}$!

One of the many virtues of this proof is that it is manifestly functorial: a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ induces a continuous linear map on the path spaces $P(\mathfrak{g}) \to P(\mathfrak{h})$, which is a homomorphism of Banach Lie groups and maps the subgroup $P(\mathfrak{g})_0$ to $P(\mathfrak{h})_0$, and therefore descends to a Lie group homomorphism.

For the details I refer you to the book; also be sure to read the historical and bibliographical notes at the end of the chapter. The global form of Lie's third theorem appears to be due to É. Cartan [12], whose first proof was based on the Levi decomposition. A later version [13] goes approximately as follows: to build a simply connected group G corresponding to $\mathfrak{g}$, start with the universal covering group G_0 of the adjoint group of $\mathfrak{g}$, and then construct G as a central extension of G_0 by the centre of $\mathfrak{g}$ (viewed as an abelian Lie group). The extension is obtained from a cocycle $G_0 \times G_0 \to \mathfrak{z}(\mathfrak{g})$, which Cartan finds by integrating the infinitesimal cocycle that corresponds to the Lie algebra extension $\mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_0$. This argument has a natural interpretation in the language of differentiable group cohomology, as was shown by Van Est [32]. Much earlier Lie himself suggested a different proof: he surmised that the Lie algebra $\mathfrak{g}$ ought to be linear and that a group with Lie algebra $\mathfrak{g}$ can therefore be realized as a subgroup of an appropriate general linear group. This line of argument was justified a few years after Cartan by Ado [1].

The point of this for Poisson geometry is that a few years after publication the Duistermaat-Kolk proof became at Alan Weinstein's suggestion a central feature of Marius Crainic and Rui Loja Fernandes' resolution of two longstanding problems in differential geometry: the integrability of a Lie algebroid to a Lie groupoid [19], and the integrability of a Poisson manifold to a symplectic groupoid [20]. Curiously, a very different work of Duistermaat, which I will get to later, also impinges on these integrability problems. My Cornell colleague Leonard Gross has a paper in preparation that adapts the Duistermaat-Kolk argument to certain infinite-dimensional situations.

Let me now discuss in a bit more detail four of Hans Duistermaat's papers that are of obvious relevance to the topics of this conference.

1. The spectrum of positive elliptic operators and periodic bicharacteristics

This paper [28], coauthored with Victor Guillemin, is Hans Duistermaat's most cited work according to MathSciNet. It is perhaps also his technically most accomplished paper. The authors consider a compact n-dimensional manifold X and a scalar elliptic pseudodifferential operator P of order 1 on X which is positive selfadjoint. The spectrum of this operator is a discrete set

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots \longrightarrow \infty.$$

By placing a Dirac measure at each eigenvalue we obtain the spectral distribution $\sigma_P = \sum_{j=0}^{\infty} \delta_{\lambda_j}$. The principal symbol p of P defines a Hamiltonian vector field H_p on the punctured cotangent bundle $T^*X \setminus X$.

Example. The main example to keep in mind is that of the Laplacian Δ defined with respect to a Riemannian metric on X. For a suitable constant c the operator $c - \Delta$ is positive selfadjoint, so the spectral theorem enables us to define a positive square root $P = \sqrt{c - \Delta}$, which is pseudodifferential of order 1 and whose principal symbol is given by $p(x, \xi) = \|\xi\|$ for $x \in X$ and $\xi \in T_xX$. The Hamiltonian flow of p is the geodesic spray of X (at least on the unit sphere bundle).

The operator P is a quantization of the classical observable p. The classical analogue of an eigenfunction of P is a relative equilibrium of the Hamiltonian vector field H_p and the classical analogue of an eigenvalue is the frequency of a periodic motion of H_p. The purpose of the paper is to make this analogy precise.

Here I must interrupt myself to state that Hans never spoke to me in such terms. Many of us conceive of mathematics as a system of grandiose functorial schemes and profound analogies or correspondences suggested by the mysteries of nature. Hans' mind worked differently and the word "quantization" never crossed his lips except in jest. Once he told me that on a visit to Moscow early in his career Gelfand asked him what were his chief mathematical goals in life, and he had no idea what to say.

What moved Hans Duistermaat, as far as I can see, was a gregarious and competitive spirit that took him from one collaboration to the next and from one mathematical problem to the next. These are the same qualities that made him a keen chess player, strong enough to have once played former world champion A. Karpov to a draw in a simultaneous match. An early manifestation of this spirit was his eager participation in the sport of kite flying during his boyhood in the Dutch East Indies, now the Republic of Indonesia. The local variant of the entertainment, which Wikipedia tells me is known as kite fighting, required coating the flying line with glass and abrasives for the purpose of ruining one's playmates' equipment. Victor Guillemin relates in his acceptance notice for the 2003 Steele Prize how Hans warned him, not for nothing, against getting involved with a *duistere maat* (murky companion).

Let us turn back to microlocal analysis and look at

$$\hat{\sigma}_P(t) = \mathscr{F}\sigma_P = \sum_{j=0}^{\infty} e^{-i\lambda_j t},$$

the Fourier transform of the spectral distribution σ_P. This can be seen as the distributional trace of the unitary operator e^{-itP}, which is a Fourier integral operator. A preliminary result says that $\hat{\sigma}_P$ is a tempered distribution, and therefore so is σ_P. In particular the eigenvalue counting function

$$N_P(\lambda) = \sharp\{\, j \mid \lambda_j \leq \lambda \,\}$$

does not grow faster than a power of λ, which foreshadows Weyl's law. The next result is a first hint at the connection between periods and eigenvalues.

Theorem. *$\hat{\sigma}_P$ is C^∞ outside the set of periods of periodic trajectories of H_p.*

In other words, the singular support of $\hat{\sigma}_P$ is contained in the set of periods of the Hamiltonian p. This is suggestive of the Poisson summation formula, where one Fourier transforms a sum of delta functions supported on a lattice and finds a sum of delta functions supported on the dual lattice. The results of Duistermaat and Guillemin describe the singularities of $\hat{\sigma}_P$ and can be viewed as a generalization of this elementary fact.

Every orbit is periodic of period 0, so one expects $\hat{\sigma}_P$ to have a big singularity at $t = 0$. To focus on this singularity take a smooth function χ such that $\hat{\chi} = \mathscr{F}\chi$ is a bump function equal to 1 in a small neighbourhood of 0. Then

$$\hat{\chi}(t)\hat{\sigma}_P(t) = \sum_{j=0}^{\infty} e^{-i\lambda_j t}\hat{\chi}(t) = \mathscr{F}\left(\sum_j \chi(\lambda - \lambda_j)\right).$$

Theorem. *We have an asymptotic expansion*

$$\sum_j \chi(\lambda - \lambda_j) \sim \frac{1}{(2\pi)^n} \sum_{k=0}^{\infty} c_k \lambda^{n-1-k}$$

as $\lambda \to \infty$. The constants c_k are independent of χ. The leading coefficient is

$$c_0 = \operatorname{vol}\{(x,\xi) \in T^*X \mid p(x,\xi) = 1\}.$$

This yields all sorts of information about the spacing of the eigenvalues, for example the following version of Weyl's law, which says that the volume of phase space is asymptotically proportional to the number of eigenvalues.

Theorem. *We have an asymptotic expansion*

$$N_P(\lambda) = \frac{a}{(2\pi)^n}\lambda^n + O(\lambda^{n-1})$$

as $\lambda \to \infty$, where $a = \operatorname{vol}\{(x,\xi) \mid p(x,\xi) \leq 1\}$.

A further analysis leads to a "residue formula", which describes the poles of $\hat{\sigma}_P$ at nonzero periods.

Theorem. *Let $T \neq 0$. Assume that all periodic orbits of H_p of period T are isolated and nondegenerate. Then*

$$\lim_{t \to T}(t-T)\hat{\sigma}_P(t) = \sum_\gamma \frac{T_{0,\gamma}}{2\pi} \frac{i^{m_\gamma}}{\left|\det(I - d\Pi_\gamma)\right|^{1/2}}.$$

There is a close resemblance between this formula and the Lefschetz formula for elliptic complexes of Atiyah and Bott [4]. The sum on the right is over all closed orbits γ of period T; $T_{0,\gamma}$ is the primitive period of γ; and Π_γ is the Poincaré return map of γ. Nondegeneracy of γ means that $\det(I - d\Pi_\gamma) \neq 0$. The integer m_γ is a Maslov index. For $P = \sqrt{c - \Delta}$ it is the Morse index of the geodesic γ for the Euler-Lagrange functional. Eckhard Meinrenken showed in an early paper [39] that for general P the number m_γ can be interpreted as a Conley-Zehnder index.

Duistermaat and Guillemin did not do this work in isolation. Some of the most important prior mathematical work on the subject is that of Weyl [46], which was inspired by Planck's model of black-body radiation, and Hörmander [37]. Roughly contemporaneous work includes that of Gutzwiller [36], Colin de Verdière [16], and Chazarain [15]. See [2] for a historical survey that takes in a good deal of the physics literature. For later developments the reader can consult the Fourier volume in honour of Colin de Verdière, particularly Colin's own contribution [17] to that volume.

2. On global action-angle variables

I will give a slightly anachronistic account of Duistermaat's paper [24] on monodromy in integrable systems, which takes into consideration later work of Dazord and Delzant [22].

Let B be a connected n-manifold. A *Lagrangian fibre bundle* over B is a triple $\mathscr{L} = (M, \omega, \pi)$, where (M, ω) is a symplectic $2n$-manifold and $\pi\colon M \to B$ a surjective submersion with Lagrangian fibres. To keep things simple we will assume that the fibres of π are compact and connected.

The standard way to obtain such a bundle is to start with an integrable Hamiltonian system and throw out the singularities of the energy-momentum map. The simplest Lagrangian fibre bundle over a given base B is as follows.

Example. Let $p\colon B \to \mathbf{R}^n$ be a local diffeomorphism. Let $\mathbf{T}$ be the circle $\mathbf{R}/\mathbf{Z}$. The *angle form* on $\mathbf{T}$ is dq, where q is the coordinate on $\mathbf{R}$. Let M be the product $B \times \mathbf{T}^n$, equipped with the symplectic form $\omega = \sum_{j=1}^n dp_j \wedge dq_j$. Let $\pi\colon M \to B$ be the projection onto the first factor. The functions p_j are the *action variables* and the (multivalued) functions q_j are the *angle variables*. The map $p \circ \pi$ is a momentum map for the translation action of $\mathbf{T}^n$ on the second factor of M.

An *isomorphism* of Lagrangian fibre bundles over B is given by a symplectomorphism of the total spaces that induces the identity map on the base. There is an equally obvious notion of *localization*, that is restriction of a Lagrangian fibre bundle to an open subset of the base. The Liouville-Mineur-Arnold theorem states that every Lagrangian fibre bundle admits local action-angle variables, i.e. is locally isomorphic to $B \times \mathbf{T}^n$. The problem solved by Duistermaat is when a Lagrangian fibre bundle over B admits global action-angle variables, i.e. is globally isomorphic to $B \times \mathbf{T}^n$.

Theorem. *A Lagrangian fibre bundle admits global action-angle variables if and only if two invariants, $\mu(\mathscr{P})$ (the affine monodromy) and $\lambda(\mathscr{L})$ (the Lagrangian class), vanish.*

Just as interesting is the fact that many commonplace integrable systems do *not* admit global action-angle variables, for instance Huygens' spherical pendulum, which Duistermaat analyses in detail. Let me now explain the two invariants.

Monodromy. Let $\mathscr{L} = (M, \omega, \pi)$ be a Lagrangian fibre bundle over B. The map $TM \to T^*M$ given by $v \mapsto \iota(v)\omega$ is a bundle isomorphism, and we denote its inverse by $\omega^\sharp\colon T^*M \to TM$. Let $m \in M$ and put $b = \pi(m) \in B$. Given a covector $\alpha \in T_b^*B$, the projection and the symplectic form produce a tangent vector $v_m(\alpha)$,

$$T_b^*B \xrightarrow{\pi^*} T_m^*M \xrightarrow{\omega^\sharp} T_mM, \qquad \alpha \longmapsto \pi^*(\alpha) \longmapsto \omega^\sharp\pi^*(\alpha) = v_m(\alpha).$$

Since we can write $\alpha = d_bf$ for a suitable function f, we see that $v_m(\alpha)$ is the value at m of the Hamiltonian vector field H_{π^*f}, and therefore is tangent to the fibre $\pi^{-1}(b)$. The fibre being compact, the vector field $v(\alpha)$ is complete, and we denote by $\varphi_b(\alpha)\colon \pi^{-1}(b) \to \pi^{-1}(b)$ its time 1 flow. The map

$$\varphi_b\colon T_b^*B \times \pi^{-1}(b) \longrightarrow \pi^{-1}(b)$$

defined by $\varphi_b(\alpha, m) = \varphi_b(\alpha)(m)$ is an action of the abelian Lie group $T_b^*B \cong \mathbf{R}^n$ on $\pi^{-1}(b)$. The map $\alpha \mapsto v_m(\alpha)$ is an isomorphism $T_b^*B \to T_m(\pi^{-1}(b))$, so, the fibre $\pi^{-1}(b)$ being connected, we conclude that the action φ_b is transitive and locally free. The kernel of the action $\mathscr{P}_b \cong \mathbf{Z}^n$ is the *period lattice* at b. Collecting these fibrewise actions gives us an action

$$\varphi\colon T^*B \times_B M \longrightarrow M$$

of the bundle of Lie groups $T^*B \to B$ on the bundle $M \to B$. The kernel of this bundle action is the bundle of free abelian groups $\mathscr{P} = \coprod_b \mathscr{P}_b$ over B, called the *period bundle*. The quotient

$$\mathscr{T} = T^*B/\mathscr{P}$$

is a bundle over B with general fibre the torus $\mathbf{T}^n$ and structure group $\mathrm{Aut}(\mathbf{T}^n) \cong \mathbf{GL}(n, \mathbf{Z})$. The quotient action

$$\varphi_{\mathscr{T}} \colon \mathscr{T} \times_B M \longrightarrow M,$$

which we will write as $\varphi_{\mathscr{T}}(t, m) = t \cdot m$, makes M a $\mathscr{T}$*-torsor*, a principal homogeneous space for the torus bundle $\mathscr{T}$ in the sense that the map $\mathscr{T} \times_B M \to M \times_B M$ defined by $(t, m) \mapsto (m, t \cdot m)$ is a diffeomorphism.

The T^*B-action defines for each 1-form on the base $\alpha \in \Omega^1(B)$ a diffeomorphism $\varphi(\alpha)$ from M to itself which induces the identity on B. This diffeomorphism transforms the symplectic form as follows.

Lemma. *$\varphi(\alpha)^*\omega = \omega + \pi^* d\alpha$ for every $\alpha \in \Omega^1(B)$.*

Recall that a Lagrangian section of a cotangent bundle is the same as a closed 1-form. Since sections of $\mathscr{P}$ induce the identity map on M, the lemma tells us therefore that $\mathscr{P}$ is a Lagrangian submanifold of T^*B.

This has various desirable consequences. First of all, applying the lemma to the translation action of T^*B on itself we conclude that the standard symplectic form is preserved by the $\mathscr{P}$-action and so descends to a symplectic form $\omega_{\mathscr{T}}$ on $\mathscr{T}$. Thus the Lie group bundle $\mathscr{T}$ itself is a Lagrangian fibre bundle over B.

More importantly, we see that on any sufficiently small open subset U of the base there exists a coordinate system $p = (p_1, p_2, \ldots, p_n)$ such that

$$\mathscr{F}(p) = (dp_1, dp_2, \ldots, dp_n)$$

is a frame of the local system $\mathscr{P}|U$. These preferred coordinate systems determine an integral affine structure on B, i.e. an atlas with values in the pseudogroup defined by the integral affine group $\mathbf{G} = \mathbf{GL}(n, \mathbf{Z}) \ltimes \mathbf{R}^n$. Conversely, this atlas determines the Lagrangian lattice bundle $\mathscr{P}$.

Analytic continuation of the coordinate system p along a loop γ in B based at $b \in U$ gives a new coordinate system p' at b, which is related to p by a transformation $g_\gamma \in \mathbf{G}$. The corresponding local frames $\mathscr{F}(p)$ and $\mathscr{F}(p')$ of $\mathscr{P}$ are related by the linear part $g_{0,\gamma} \in \mathbf{G}_0 = \mathbf{GL}(n, \mathbf{Z})$ of the affine transformation g_γ. The map $\gamma \mapsto g_\gamma$ induces a homomorphism from $\pi_1(B, b)$ to $\mathbf{G}$. The conjugacy class of this homomorphism,

$$\mu(\mathscr{P}) \in \mathrm{Hom}(\pi_1(B), \mathbf{G}) / \mathrm{Ad}(\mathbf{G}) \cong H^1(B, \mathbf{G}),$$

is the *affine monodromy* of $\mathscr{P}$. (Here $H^1(B, \mathbf{G})$ denotes the cohomology set of B with coefficients in the locally constant sheaf $\mathbf{G}$.) The conjugacy class defined by the map $\gamma \mapsto g_{0,\gamma}$,

$$\mu_0(\mathscr{P}) \in \mathrm{Hom}(\pi_1(B), \mathbf{G}_0) / \mathrm{Ad}(\mathbf{G}_0) \cong H^1(B, \mathbf{G}_0),$$

is the *linear monodromy*, which determines the isomorphism class of the local system $\mathscr{P}$. The monodromy depends only on the affine structure of B, not on M or its symplectic structure.

The class $\mu_0(\mathscr{P})$ vanishes if and only if $\mathscr{P}$ is trivial. In that case $\mathscr{T} \cong B \times \mathbf{T}^n$ is isomorphic to a trivial bundle of Lie groups, M is a principal $\mathbf{T}^n$-bundle over B, and $\mathscr{P}$ has a global frame of closed 1-forms $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. We can then find a covering $f \colon \tilde{B} \to B$ of the base and a local diffeomorphism $\tilde{p} \colon \tilde{B} \to \mathbf{R}^n$ such that $f^*\alpha_j = d\tilde{p}_j$. If the α_j are exact, then the full monodromy $\mu(\mathscr{P})$ vanishes, we can define global single-valued action variables $p\colon B \to \mathbf{R}^n$, and $p \circ \pi$ is a momentum map for the $\mathbf{T}^n$-action on M.

Chern class and Lagrangian class. The existence of global angle variables on a Lagrangian fibre bundle $\mathscr{L} = (M, \omega, \pi)$ is tantamount to the existence of a global Lagrangian section of $\pi\colon M \to B$.

First let us consider plain smooth sections of π. We need to introduce a few sheaves of abelian groups on the base space B. There is Ω^k, the sheaf of smooth k-forms, and its subsheaf $\mathscr{Z}^k$ of closed k-forms. Then there is the sheaf of smooth sections of $\mathscr{T}$, which we will simply call $\mathscr{T}$, and the sheaf of locally constant sections of $\mathscr{P}$, which we will likewise simply call $\mathscr{P}$. Let $\{U_i\}_{i\in I}$ be an open cover of B and suppose that we have local smooth sections $s_i\colon U_i \to M$ of π. Since M is a $\mathscr{T}$-torsor, over each intersection $U_{ij} = U_i \cap U_j$ we have a unique section $t_{ij} \in \mathscr{T}(U_{ij})$ such that $s_i = t_{ij} \cdot s_j$. The tuple $t = (t_{ij})$ is a Čech 1-cocycle and defines an element $[t] \in H^1(B, \mathscr{T})$.

Since $\mathscr{T}$ is the quotient bundle $T^*B/\mathscr{P}$, on the level of sheaves we have a short exact sequence

$$0 \longrightarrow \mathscr{P} \longrightarrow \Omega^1 \longrightarrow \mathscr{T} \longrightarrow 0.$$

The sheaf Ω^1 is fine, so the long exact cohomology sequence gives canonical isomorphisms

$$H^k(B, \mathscr{T}) \cong H^{k+1}(B, \mathscr{P})$$

for all $k \geq 0$. The image $c(M) \in H^2(B, \mathscr{P})$ of $[t]$ is the *Chern class* of the $\mathscr{T}$-torsor M and it is the obstruction to the existence of a global section of π. It is independent of the symplectic structure on M.

Since $\mathscr{P}$ is Lagrangian, it is a subsheaf of $\mathscr{Z}^1$, and therefore the exterior derivative $d\colon \Omega^1 \to \mathscr{Z}^2$ descends to a morphism

$$d_{\mathscr{P}}\colon \mathscr{T} \longrightarrow \mathscr{Z}^2.$$

A section t of $\mathscr{T}$ is *closed* if $d_{\mathscr{P}} t = 0$. If the open sets U_i are small enough, we can choose the local sections s_i to be Lagrangian, which implies that the transition functions t_{ij} are closed. Thus the t_{ij} are sections of the subsheaf $\mathscr{K} = \ker d_{\mathscr{P}}$ of $\mathscr{T}$, and the corresponding cohomology class lives in $H^1(B, \mathscr{K})$. This is the *Lagrangian class* $\lambda(\mathscr{L})$, which is implicit in the paper of Dazord and Delzant but was named by Zung [49], and it is the obstruction to the existence of a global *Lagrangian* section of π. Given a Lagrangian section s, the map $T^*B \to M$ defined by $(b, \alpha) \mapsto \varphi(\alpha)(s(b))$ identifies the Lagrangian fibre bundle $\mathscr{T}$ with $\mathscr{L}$.

Therefore the vanishing of the Lagrangian class $\lambda(\mathscr{L})$ is equivalent to $\mathscr{L}$ being isomorphic as a Lagrangian fibre bundle to $\mathscr{T}$. The vanishing of both $\lambda(\mathscr{L})$ and the affine monodromy $\mu(\mathscr{P})$ is equivalent to the existence of global action-angle variables. This is the version of Duistermaat's theorem established by Dazord and Delzant (who, by the way, also considered the case of less than fully integrable systems).

Symplectic torsors. Dazord and Delzant went on to prove that the monodromy and the Lagrangian class completely classify all Lagrangian fibre bundles. Let us widen our view a little by fixing an integral affine manifold B with period bundle $\mathscr{P}$ and torus bundle $\mathscr{T} = T^*B/\mathscr{P}$, and examining arbitrary $\mathscr{T}$-torsors over B. Any such torsor $\pi\colon M \to B$ has a well-defined Chern class $c(M) \in H^2(B, \mathscr{P})$.

In fact, just as for principal bundles the cohomology group $H^2(B, \mathscr{P})$ classifies $\mathscr{T}$-torsors up to isomorphism. (If the linear monodromy $\mu_0(\mathscr{P})$ vanishes, then $\mathscr{P}$ is the constant local system $\mathbf{Z}^n$, a $\mathscr{T}$-torsor is an ordinary principal $\mathbf{T}^n$-bundle, and the Chern class is the ordinary Chern class in $H^2(B, \mathbf{Z}^n)$.)

Let us think about all possible symplectic forms ω on M which vanish on the fibres of π, so making $\mathscr{L} = (M, \omega, \pi)$ into a Lagrangian fibre bundle. We will call $\mathscr{L}$ a *symplectic $\mathscr{T}$-torsor* with *total space M*. As before we regard two symplectic $\mathscr{T}$-torsors as isomorphic if the total spaces are symplectomorphic via a diffeomorphism that fixes the base B. The collection of isomorphism classes $[\mathscr{L}]$ is an analogue of the Picard group of an algebraic variety and we will denote it by $\mathbf{Pic}(B, \mathscr{P})$.

This set is equipped with two algebraic operations. The *opposite* of $\mathscr{L} = (M, \omega, \pi)$ is $-\mathscr{L} = (M, -\omega, \pi)$. (Negating the almost symplectic form has the effect of reversing the $\mathscr{T}$-action, i.e. composing it with the automorphism $t \mapsto t^{-1}$ of $\mathscr{T}$.) Given two symplectic $\mathscr{T}$-torsors $\mathscr{L}_1 = (M_1, \omega_1, \pi_1)$ and $\mathscr{L}_2 = (M_2, \omega_2, \pi_2)$, define M to be the $\mathscr{T}$-torsor $(M_1 \times_B M_2)/\mathscr{T}^-$, where $\mathscr{T}^-$ is the antidiagonal subbundle $\{(t, t^{-1}) \mid t \in \mathscr{T}\}$ of $\mathscr{T} \times_B \mathscr{T}$. It is a theorem of Ping Xu [48] that the form $\omega_1 + \omega_2$ on $M_1 \times_B M_2$ descends to an almost symplectic form ω on M which makes $\mathscr{L} = (M, \omega, \pi)$ into a symplectic $\mathscr{T}$-torsor. We call $\mathscr{L}$ the *sum* of $\mathscr{L}_1$ and $\mathscr{L}_2$. The operation $[\mathscr{L}_1] + [\mathscr{L}_2] = [\mathscr{L}_1 + \mathscr{L}_2]$ turns $\mathbf{Pic}(B, \mathscr{P})$ into an abelian group. The zero element is $[\mathscr{T}]$ and the opposite of $[\mathscr{L}]$ is $[-\mathscr{L}]$.

Can we explicitly describe the Picard group $\mathbf{Pic}(B, \mathscr{P})$? The Poincaré lemma implies that

$$0 \longrightarrow \mathscr{Z}^k \longrightarrow \Omega^k \xrightarrow{d} \mathscr{Z}^{k+1} \longrightarrow 0$$

is a short exact sequence of sheaves. To begin with, this gives us isomorphisms

$$H^l(B, \mathscr{Z}^k) \cong H^{k+l}(B, \mathbf{R}),$$

because Ω^k is fine. Furthermore, taking $k = 1$ and dividing the first two terms by $\mathscr{P}$ we get the short exact sequence

$$0 \longrightarrow \mathscr{Z}^1/\mathscr{P} \longrightarrow \mathscr{T} \xrightarrow{d_{\mathscr{P}}} \mathscr{Z}^2 \longrightarrow 0.$$

This identifies the kernel $\mathscr{K} = \ker d_{\mathscr{P}}$ with $\mathscr{Z}^1/\mathscr{P}$ and yields a long exact sequence

$$0 \longrightarrow H^0(B, \mathscr{K}) \longrightarrow H^0(B, \mathscr{T}) \xrightarrow{d_{\mathscr{P},*}} H^0(B, \mathscr{Z}^2) \xrightarrow{\partial} H^1(B, \mathscr{K}) \longrightarrow H^1(B, \mathscr{T})$$
$$\xrightarrow{d_{\mathscr{P},*}} H^1(B, \mathscr{Z}^2) \xrightarrow{\partial} H^2(B, \mathscr{K}) \longrightarrow \cdots$$

Substituting $H^k(B, \mathscr{Z}^2) \cong H^{k+2}(B, \mathbf{R})$ and $H^k(B, \mathscr{T}) \cong H^{k+1}(B, \mathscr{P})$, and noticing that $H^k(B, \mathscr{K}) \to H^k(B, \mathscr{T}) \cong H^{k+1}(B, \mathscr{P})$ is the connecting homomorphism δ for the short exact sequence

$$0 \longrightarrow \mathscr{P} \longrightarrow \mathscr{Z}^1 \longrightarrow \mathscr{K} \longrightarrow 0,$$

we obtain the long exact sequence that we want,

$$0 \longrightarrow H^0(B, \mathscr{K}) \xrightarrow{\delta} H^1(B, \mathscr{P}) \xrightarrow{d_{\mathscr{P},*}} H^2(B, \mathbf{R}) \xrightarrow{\partial} H^1(B, \mathscr{K}) \xrightarrow{\delta} H^2(B, \mathscr{P})$$
$$\xrightarrow{d_{\mathscr{P},*}} H^3(B, \mathbf{R}) \xrightarrow{\partial} H^2(B, \mathscr{K}) \xrightarrow{\delta} \cdots$$

If a $\mathscr{T}$-torsor M admits a symplectic form ω vanishing on the fibres, then δ maps the Lagrangian class $\lambda(M, \omega, \pi)$ to the Chern class $c(M)$, and therefore $d_{\mathscr{P},*}c(M) = 0$. So we see that $d_{\mathscr{P},*}c(M) = 0$ is a necessary condition for M to be the total space of a symplectic $\mathscr{T}$-torsor. Dazord and Delzant show that this condition is actually sufficient, and that every $\lambda \in H^1(B, \mathscr{K})$ satisfying $\delta\lambda = c(M)$ is the Lagrangian class of a unique isomorphism class of Lagrangian fibre bundles with total space M. The conclusion is as follows.

Theorem. *Let B be an integral affine manifold with period bundle $\mathscr{P}$. Let $\mathscr{T}$ be the torus bundle $T^*B/\mathscr{P}$ and let $\mathscr{K}$ be the kernel of the sheaf homomorphism $d_{\mathscr{P}} \colon \mathscr{T} \to \mathscr{Z}^2$.*

(i) *The map $\mathbf{Pic}(B, \mathscr{P}) \to H^1(B, \mathscr{K})$ defined by $[\mathscr{L}] \mapsto \lambda(\mathscr{L})$ is a group isomorphism.*

(ii) *We have a short exact sequence*

$$0 \longrightarrow H^2(B, \mathbf{R})/d_{\mathscr{P},*}H^1(B, \mathscr{P}) \longrightarrow \mathbf{Pic}(B, \mathscr{P}) \longrightarrow \ker(d_{\mathscr{P},*}) \longrightarrow 0.$$

This theorem gives us two different descriptions of the identity component of the Picard group, namely $\mathbf{Pic}^0(B, \mathscr{P})$ is equal to the group of symplectic torsors of "degree" (i.e. Chern class) 0, and

$$\mathbf{Pic}^0(B, \mathscr{P}) \cong H^2(B, \mathbf{R})/d_{\mathscr{P},*}H^1(B, \mathscr{P}).$$

The "Néron-Severi group" (i.e. component group) $\mathbf{Pic}(B, \mathscr{P})/\mathbf{Pic}^0(B, \mathscr{P})$ is isomorphic to the subgroup $\ker(d_{\mathscr{P},*})$ of $H^2(B, \mathscr{P})$. If the base B is of finite type, the Picard group is finite-dimensional and the Néron-Severi group is finitely generated.

Suppose that we are given a $\mathscr{T}$-torsor $\pi \colon M \to B$ with Chern class $c \in H^2(B, \mathscr{P})$ and let us denote by $\mathbf{Pic}(M, B, \mathscr{P}) \cong \delta^{-1}(c)$ the collection of isomorphism classes of symplectic $\mathscr{T}$-torsors with total space M. The theorem tells us that $\mathbf{Pic}(M, B, \mathscr{P})$ is nonempty if and only if $d_{\mathscr{P},*}c = 0$ and that the group $\mathbf{Pic}^0(B, \mathscr{P})$ acts simply transitively on $\mathbf{Pic}(M, B, \mathscr{P})$. The action is the *gauge action* given by the formula $[\sigma] \cdot [M, \omega, \pi] = [M, \omega + \pi^*\sigma, \pi]$, where $\sigma \in Z^2(B)$ is a de Rham representative of a class in $H^2(B, \mathbf{R})$.

Zung has obtained a version of these results for certain *singular* Lagrangian fibrations.

Twisted symplectic torsors. It is instructive to go one step further in the long exact sequence and ask what happens if $d_{\mathscr{P},*}c(M)$ is nonzero. This leads to a "nonholonomic" or "quasi-Hamiltonian" version of the Duistermaat-Dazord-Delzant theorems. I will outline the results and publish the proofs elsewhere. We define a *twisted Lagrangian fibre bundle* $\mathscr{L} = (M, \omega, \pi)$ over a base manifold B in the same way as a Lagrangian fibre bundle, except that we drop the requirement that ω be closed. Thus ω is an almost symplectic form.

It turns out that, just as in the Lagrangian case, the cotangent bundle T^*B acts on the total space M of a twisted Lagrangian fibre bundle $\mathscr{L}$ and that the kernel of the action is a bundle of lattices $\mathscr{P}$, which is Lagrangian with respect to the standard symplectic form on T^*B. So again B is an integral affine manifold and M is a torsor for the torus bundle $\mathscr{T} = T^*B/\mathscr{P}$.

We now fix the integral affine manifold $(B, \mathscr{P})$ and look at any twisted Lagrangian bundle $\mathscr{L} = (M, \omega, \pi)$ which is at the same time a $\mathscr{T}$-torsor. We assume that the almost symplectic form ω is *compatible* with the $\mathscr{T}$-action in the sense that the T^*B-action on M induced by ω has kernel $\mathscr{P}$. We call such an $\mathscr{L}$ a *twisted symplectic $\mathscr{T}$-torsor* and set ourselves the task of classifying up to isomorphism all twisted symplectic $\mathscr{T}$-torsors. We denote the set of isomorphism classes by $\mathbf{TPic}(B, \mathscr{P})$.

The first observation is that this set is an abelian group in the same way as the ordinary Picard group. We will refer to $\mathbf{TPic}(B, \mathscr{P})$ as the *twisted* Picard group of the integral affine manifold $(B, \mathscr{P})$.

The next observation is that every $\mathscr{T}$-torsor M possesses a compatible almost symplectic form and that the extent to which this form is not closed is measured by the class $d_{\mathscr{P},*}c(M)$.

Theorem. *Every $\mathscr{T}$-torsor $\pi\colon M \to B$ possesses a compatible almost symplectic form ω. Every such form ω satisfies $d\omega = \pi^*\eta$ for a unique closed 3-form $\eta \in Z^3(B)$. We have $[\eta] = d_{\mathscr{P},*}c(M)$.*

We call the closed 3-form $\eta = \eta(\mathscr{L})$ the *twisting form* of the twisted Lagrangian fibre bundle $\mathscr{L}$. It is an isomorphism invariant of $\mathscr{L}$. The pair (ω, η) is a cocycle in the relative de Rham complex of the projection π.

The Dazord-Delzant theorem generalizes as follows.

Theorem. *We have an exact sequence*

$$0 \longrightarrow \Omega^2(B)/d_{\mathscr{P}}H^0(B, \mathscr{T}) \longrightarrow \mathbf{TPic}(B, \mathscr{P}) \longrightarrow H^2(B, \mathscr{P}) \longrightarrow 0.$$

So the moduli space of twisted Lagrangian fibre bundles is typically infinite-dimensional. These degrees of freedom can be taken away by introducing a coarser form of gauge equivalence, namely by letting an arbitrary 2-form $\sigma \in \Omega^2(B)$ on the base act on a twisted Lagrangian fibre bundle $\mathscr{L} = (M, \omega, \pi)$ by the formula $\sigma \cdot \mathscr{L} = (M, \omega + \pi^*\sigma, \pi)$. This action changes the twisting form by the exact 3-form $d\beta$. It follows from the theorem that $\mathbf{TPic}(B, \mathscr{P})/\Omega^2(B)$ is isomorphic to $H^2(B, \mathscr{P})$, in other words every $\mathscr{T}$-torsor has a compatible almost symplectic structure which is unique up to coarse gauge equivalence.

A twisted symplectic $\mathscr{T}$-torsor does not have a well-defined Lagrangian class, but the *difference* $\mathscr{L}_1 - \mathscr{L}_2 = \mathscr{L}_1 + -\mathscr{L}_2$ of two twisted symplectic $\mathscr{T}$-torsors that have the same twisting forms, $\eta(\mathscr{L}_1) = \eta(\mathscr{L}_2)$, is a symplectic $\mathscr{T}$-torsor and therefore has a well-defined Lagrangian class. It follows that if we fix a closed 3-form $\eta \in Z^3(B)$ the set of isomorphism classes of twisted symplectic $\mathscr{T}$-torsors with twisting form η is a principal homogeneous space of $\mathbf{Pic}(B, \mathscr{P})$. If in addition we fix a class $c \in H^2(B, \mathscr{P})$ satisfying $d_{\mathscr{P},*}c = [\eta]$, then the set of isomorphism classes of twisted symplectic $\mathscr{T}$-torsors with Chern class c and twisting form η is a principal homogeneous space of $\mathbf{Pic}^0(B, \mathscr{P})$.

Groupoids and realizations. At the end of my talk Alan Weinstein pointed out that Duistermaat's study of global action-angle variables provided one of the incentives for him to formulate the symplectic groupoid program [18], [45]. In the language of that program a Lagrangian fibre bundle is nothing but a *realization* of the base manifold B equipped with the zero Poisson structure, and the torus bundle $\mathscr{T}$ is a symplectic groupoid over B (with source and target maps being equal) which *integrates* this Poisson manifold.

Every manifold B with zero Poisson structure is obviously integrable and the associated source-simply connected symplectic groupoid is just the cotangent bundle T^*B. What makes the integral affine case special is the existence of a *proper* symplectic groupoid $\mathscr{T}$ which integrates the trivial Poisson structure. The Duistermaat-Dazord-Delzant theorems then amount to a classification of all realizations of B which are free and fibre-transitive under $\mathscr{T}$. The group $\mathbf{Pic}(B, \mathscr{P})$ is referred to as the *static* Picard group of the groupoid $\mathscr{T}$ in [11]. (The full, noncommutative, Picard group is the semidirect product of $\mathbf{Pic}(B, \mathscr{P})$ with the group of integral affine automorphisms of B.) The twisted case also fits into this framework, as one can see from the papers [10] and [14].

The quantum mechanical spherical pendulum. Having spent far more time on action-angle variables than I intended, let me be very brief about the quantum mechanical picture. A treatment of quantum monodromy in the spherical pendulum was given by Richard Cushman and Hans Duistermaat [21]. A different interpretation was given soon afterwards by Victor Guillemin and Alejandro Uribe [35]. Let me quote from Hans' review of the latter paper in the *Mathematical Reviews*, which he starts by explaining his own approach:

> If one considers the Schrödinger operator $E = -(\hbar^2/2)\Delta + V$, where Δ is the Laplace operator on the 2-dimensional standard sphere S in $\mathbf{R}^3$ and the potential V is the vertical coordinate function, then the rotational symmetry around the vertical axis yields an operator $L = i\hbar(x_1\partial/\partial x_2 - x_2\partial/\partial x_1)$ which commutes with S [*sic*].
>
> Replacing $i\hbar\partial/\partial x_j$ by the conjugate variable p_j, we get principal symbols e and l of E and L, respectively, which Poisson commute and define an integrable Hamiltonian system on the phase space T^*S, the cotangent bundle of S. One has straightforward generalizations to n commuting operators $E_1, \ldots, E_n$ with principal symbols $e_1, \ldots, e_n$ on n-dimensional manifolds M.
>
> Because the operators E_j commute, one has common eigenfunctions ψ_k, $k = 1, 2, \ldots$, with eigenvalues $\varepsilon_{j,k}$ ($E_j\psi_k = \varepsilon_{j,k} \cdot \psi_k$). The rule for finding the n-dimensional spectrum $(\varepsilon_{1,k}, \ldots, \varepsilon_{n,k}) \in \mathbf{R}^n$ for $k = 1, 2, \ldots$, asymptotically for $\hbar \downarrow 0$ and near a regular value of the mapping $(e_1, \cdots, e_n)$ is as follows. One constructs locally so-called action variables, which are functions $(a_1, \ldots, a_n)$ of the $(e_1, \ldots, e_n)$, in such a way that $(\partial a_i/\partial e_j)$ is invertible and the Hamiltonian flows of the a_j are periodic with period 2π. Then the n-dimensional spectrum is given asymptotically by $a^{-1}(\mathbf{Z}^n + \alpha)$, where $\mathbf{Z}^n$ is the integer lattice, α is a Maslov shift, and a is the vector of action variables given above. This means that the actions, in particular the nonexistence of global action variables, can be read off from the asymptotics of the spectrum.

He proceeds to explain the different approach taken by Guillemin and Uribe. Although it has some very convincing illustrations, Cushman and Duistermaat's paper is little more than an announcement and there does not seem to exist a more comprehensive version. Some ten years after its appearance experimental evidence of quantum monodromy was found and finally Vũ Ngọc San wrote two papers [43], [44] clarifying and elaborating on Cushman and Duistermaat's ideas.

3. Duistermaat-Heckman

Of all Hans Duistermaat's accomplishments the best known to differential geometers is probably the Duistermaat-Heckman theorem [29]. This is so familiar to most of the audience that I passed it over in my talk, but in this written version I can't resist making some remarks about it. Recall that in its simplest form the theorem states that

$$\int_M \exp(\omega - tf) = \int_X \frac{\exp(\omega - tf)}{\mathbf{e}(X,t)}.$$

Here (M, ω) is a compact symplectic manifold, t is a complex parameter, f is a periodic Hamiltonian, X is the critical manifold of f, and $\mathbf{e}(X,t)$ is the equivariant Euler class of the normal bundle of X in M. The integral on the left is to be interpreted as the integral of $e^{-tf}\omega^n/n!$, where $2n = \dim M$. This is precisely the Fourier-Laplace transform of the measure $f_*(\omega^n/n!)$ obtained by pushing forward the Liouville measure $\omega^n/n!$ to the real line. The critical manifold X usually consists of connected components of various dimensions, so the integral on the right is to be read as a sum of integrals, one for each component.

The theorem contains as a special case Archimedes' result that the surface area of a sphere is equal to that of the circumscribed cylinder, an illustration of which, according to Cicero [42, Liber V, §§ 64–66], adorned the Syracusan's tomb. A modern antecedent of the theorem is Bott's residue formula for holomorphic vector fields [8]. Soon after publication three interesting alternative proofs appeared, one based on the localization principle in equivariant cohomology by Atiyah and Bott [5] and Berline and Vergne [7] (see also [6] and [34]), one based on partial action-angle variables by Dazord and Delzant [22], and one based on the coisotropic embedding theorem by Guillemin and Sternberg [33].

The index theorem. My favourite meta-application of the Duistermaat-Heckman theorem is Atiyah's heuristic derivation [3] of the Atiyah-Singer index theorem for the Dirac operator suggested by ideas of Witten [47]. Let X be a compact Riemannian manifold and let $M = C^\infty(S^1, X)$ be the loop space of X. A tangent vector to M at a loop γ is a vector field along γ, i.e. a section of $\gamma^*(TX)$. The loop space has a Riemannian structure: the inner product of two tangent vectors s_1, $s_2 \in T_\gamma M$ is defined to be the integral $\int_{S^1}(s_1(\theta), s_2(\theta))\,d\theta$. The circle S^1 acts on M by spinning the loops, and we let α be the 1-form on M dual to the infinitesimal generator of this action. Then $\omega = d\alpha$ is a presymplectic structure on M; it degenerates for example at the closed geodesics of X.

Despite this degeneracy the circle action is generated by a Hamiltonian, namely the energy function $E\colon M \to \mathbf{R}$ given by $E(\gamma) = \frac{1}{2}\int_{S^1}\|d\gamma\|^2$. The Duistermaat-Heckman theorem tells us to integrate the functional e^{-tE} times a "Liouville" volume form on M. The "Riemannian" volume form on M is the Wiener measure $d\gamma$, and just as in the finite-dimensional case we must multiply this by the (regularized) Pfaffian of the skew symmetric endomorphism of TM defined by the presymplectic form. This Pfaffian exists if the manifold X has a **Spin**-structure, and the Duistermaat-Heckman integral $\int_M e^{-tE(\gamma)}\,\mathrm{Pf}(\omega)\,d\gamma$ is seen to be the index of the associated Dirac operator $\eth$. The Duistermaat-Heckman theorem then says that this integral localizes to the fixed point set M^{S^1}, which is a copy of X. By calculating the weights of the action on the normal bundle of X in M one arrives at the A-roof genus and thus concludes that $\mathrm{index}(\eth) = \hat{A}(X)$.

Nitta's theorem. Is there a generalization of the Duistermaat-Heckman theorem to Poisson manifolds? In general this seems too much to ask for, but a reasonable compromise was found by Yasufumi Nitta [40, 41]. I state his result in a more general form obtained by (my student and Hans' grand-student) Yi Lin [38]. Let

(M, ρ) be a compact *generalized Calabi-Yau manifold*, that is a $2n$-dimensional manifold equipped with a (possibly twisted) generalized complex structure defined by a pure spinor $\rho \in \Omega^*(M, \mathbf{C})$ with the property that the $2n$-form $\nu = (\rho, \bar{\rho})$ is a volume form. (Such manifolds are discussed in more detail in Gil Cavalcanti's lecture notes in these proceedings.) Let T be a torus acting on M in a Hamiltonian fashion with generalized moment map $\Phi\colon M \to \mathfrak{t}^*$. Then the pushforward measure $\Phi_*(\nu)$ on $\mathfrak{t}^*$ is equal to a piecewise polynomial function times Lebesgue measure. For a symplectic manifold (M, ω) we have $\rho = e^{i\omega}$ and $\nu = \omega^n/n!$, and so this assertion is exactly the Fourier transformed version of the classical Duistermaat-Heckman theorem.

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